Real quadratic form ⇒

An expression of the form $\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} x_i x_j$, where a_{ij} are real and $a_{ij} = a_{ji}$, is said to be a real quadratic form in n variables $x_1, x_2, \ldots, x_n$.

The matrix notation for the quadratic form is $X^t A X$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

and $A = (a_{ij})_{n \times n}$

A is real symmetric matrix since $a_{ij} \in \mathbb{R}$ and $a_{ij} = a_{ji}$.

To every real quadratic form in n variables is associated an $n \times n$ real symmetric matrix which is said to be the matrix of the quadratic form.

Examples ⇒

1) $5x_1^2 + 2x_1x_2 - x_2^2$ is a real quadratic form in two variables x_1, x_2.

The associated matrix is $\begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}$.

2) $x_1x_2 - x_2x_3$ is a real quadratic form in three variables, x_1, x_2, x_3

The associated matrix is $\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}$

3) $x_1^2 - x_2^2 + 2x_3^2$ is a real quadratic form in three variables.

The associated matrix is the diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Definitions $\Rightarrow$ A real quadratic form $Q(x_1, x_2, ---, x_n)$ assumes the value 0 when $X = O$. But Q takes different real values for different non-zero X.

A real quadratic form $Q = X^tAX$ is said to be

i) positive definite if $Q > 0$, $\forall\, X \neq O$

ii) positive semidefinite if $Q \geq 0$ $\forall\, X$ and $Q = 0$ for some $X \neq$

iii) Negative definite if $Q < 0 \ \forall X \neq 0$.

iv) Negative semi definite if $Q \leq 0, \forall X$ and $Q = 0$ for some $X \neq 0$.

v) Indefinite if $Q \geq 0$ for some $X \neq 0$ and $Q \leq 0$ for some othere $X \neq 0$.

Examples ⇒

1) Let $Q = Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3x_1$.

$Q = (x_1 + x_2 - x_3)^2 + (x_2 - x_3)^2 + 2x_3^2$

≥ 0 for all $X = (x_1, x_2, x_3)$

and $Q = 0$ only when $x_1 = x_2 = x_3 = 0$.

Therefore Q is positive definite.

The associated matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix}$$ is therefore positive definite.

2) Let $Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 - 4x_2x_3 - 2x_3x_1$

$\Rightarrow Q = (x_1 + x_2 - x_3)^2 + (x_2 - x_3)^2$.

$\Rightarrow Q \geq 0$ for all $X = (x_1, x_2, x_3)$.

But Q may be zero for non-zero (x_1, x_2, x_3). For example, $Q(0,1,1) = 0$.

Therefore Q is positive semi-definite.

3) Let $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1$

$\Rightarrow Q = (x_1 + x_2 + x_3)^2 - x_3^2$

Q is indefinite because $Q > 0$ for some $X = (x_1, x_2, x_3)$ and $Q < 0$ for some X.

For example

$$Q(1,1,0) > 0$$

$$Q(0,-1,1) < 0.$$

By the transformation $X = PY$ where $|P| \neq 0$ the real quadratic form $Q = X^t A X$ transforms to $Y^t(P^t A P)Y$. This is a quadratic form in Y. [Here A is symmetric]

Since $P^t A P$ is a symmetric matric then Q is transformed to a quadratic form Q'. And Q' is said to be congruent to Q.

For a real symmetric matrix A of ~~rand~~ rank $r (\le n)$, $\exists$ a non-singular matrix P such that $P^t A P$ becomes a diagonal matrix [signature $= 2m - r$]

$$D = \begin{bmatrix} I_m & & \\ & -I_{r-m} & \\ & & 0 \end{bmatrix}$$ of rank r

where $0 \le m \le r$

Therefore by a suitable transformation $X = PY$, where P is non-singular.
The real quadratic form Q transforms to $y_1^2 + y_2^2 + \cdots + y_m^2 - y_{m+1}^2 - y_{m+2}^2 - \cdots - y_r^2$

where $0 \le m \le r \le n$.

This is said to be the normal (diagonal) form of Q.

Theorem ⇒ The integer m which is the number of positive elements in the normal (diagonal) form of a real quadratic form Q, is invariant.

Definitions The rank of a real quadratic form is defined to be the rank of associated real symmetric matrix.

The signature of a real quadratic form is defined to be the signature of the associated real symmetric matrix.